Logarithms and Exponential Growth/Decay

Math 130 - Essentials of Calculus

24 March 2021

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Definition

Let $b > 0$ be a real number with $b \neq 1$. The logarithm with base b is the function

$$f(x) = \log_b x$$

which satisfies the property

$$\log_b x = y \iff b^y = x.$$

When the base of the logarithm is the natural number e, we write $\ln x$ *instead and call it the natural logarithm.*

Example - Computing Logarithms - No Calculators!

Example

Find the exact value of the following

1. $\log_2 64$

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3. $\log_8 2$
4. $\ln e^3$

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5. $e^{\ln 4}$

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7. $\ln 1$

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7. $\ln 1$
8. $\ln e^x$

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8. $\ln e^x$
9. $e^{\ln x}$

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The only function type that we've covered and not yet taken the derivative of is a logarithmic function. We can accomplish this derivative by using implicit differentiation and rewriting $y = \ln x$.

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In this case, we'd really like an expression for the derivative that has only x in it, and since we know $e^y = x$, just plug that in to get

$$\frac{dy}{dx} = \frac{1}{x}.$$

Examples

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Find the derivative of the given functions

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1. $y = 3x - 2\ln x$
2. $f(x) = \ln x^2$
3. $y = \sqrt[5]{\ln x}$
4. $y = x\ln(1 + e^x)$

Constant Percentage Growth/Decay

Suppose you have an investment of \$1000 that grows at 5% per year, then after 1 year it is worth

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Similarly, it is possible to look at something whose value shrinks at a constant percentage: for example, say the value of a \$20000 car decreased by 15% per year, then the value after the first year would be

$$\$20000(1 - .15) = \$20000(0.85)$$

and after t years it would be worth

$$\$20000(0.85)^t$$

Example

Example

A rare sculpture was purchased for $\$11.8$ *million and its value is expected to increase* 14% *per year.*

1. *Write an equation for the function that gives the value of the sculpture after* t *years.*
2. *What is the value of the sculpture after* 3.25 *years?*
3. *After how many years will the sculpture be worth* $\$20$ *million?*

Now You Try It!

Example

The eagle population in a state park is currently 1650 but is expected to decrease 18% per year.

1. *Write an equation for the function that gives the number of eagles in the park t years from now.*
2. *Determine the time required for the population to be reduced to 1000.*
3. *What is the rate of change of the population after four years?*

Compound Interest

If a savings account earns an annual interest rate of r (expressed as a decimal, not a percentage), then the future value of the account after t years with an initial investment of P dollars would be

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More typically, you will have a compounding period of less than a year, such as monthly or quarterly. If the compounding happens n times per year (e.g., $n = 4$ for quarterly compounding), then the interest rate per quarter will be given by $\frac{r}{n}$, where r is still given as a yearly rate. In this case, the future value after t years will be

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Compounding Continuously

It is also possible to increase the compounding to happen at every instant of time, which would correspond to taking the limit as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} A(t) &= \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} \\
&= \lim_{n\to\infty} P\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} \\
&= P\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} \\
&= P\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{rt} \\
&= Pe^{rt}
\end{aligned}$$

Example

Example

If $3000 is invested at 5% interest, find the value of the investment if interest is compounded

1. *annually*
2. *quarterly*
3. *monthly*
4. *continuously*

How long will it take for the value of the investment to double if the interest is compounded quarterly?